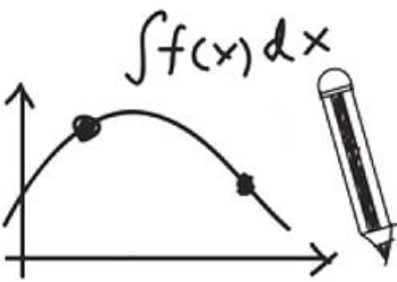


Calculus(I)

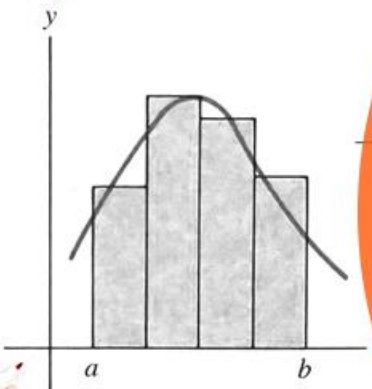
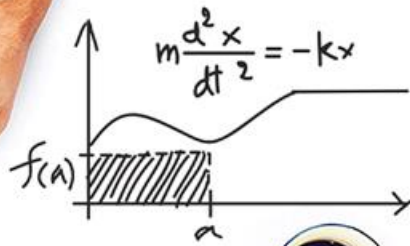
$$x^2 - 3x - 4 = 0$$

$$4x^2 - 3x - 1 = 0$$



$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$F = mg = ma = m \frac{d^2h}{dt^2}$$



Gottfried Wilhelm Leibniz

$$\frac{dA}{dt} = \frac{dB}{dt} = -\frac{dC}{dt} = \frac{dD}{dt} = (c_1)T^{\frac{1}{2}}AB - (c_2)T^{\frac{1}{2}}CD$$

$$m \frac{d^2x}{dt^2} = -kx - f \frac{dx}{dt} + A \sin(\omega t)$$

$$y' = \text{and } v' = -ky - fv + A \sin(\omega t)$$

$$m \frac{d^2x}{dt^2} = -kx$$

$$x = A \frac{dT}{dt} - (c_1)(T - T)$$



$$\frac{df(x)}{dx}$$

$$\frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$



$$x + h, f(x + \tau)$$



2.5 The Chain Rule

Lecturer: Xue Deng

Problem Introduction



How to find the derivatives of the composite functions?

$$f(x) = \ln \sin x \quad \text{and} \quad f(x) = e^{\sin \frac{1}{x}}$$

Theorem of Chain Rule

Let $y = f(u)$ and $u = g(x)$

If g is differentiable at x and f is differentiable at $u = g(x)$

then the composite function $f \circ g$, defined by $(f \circ g)(x) = f(g(x))$,

is differentiable at x and $(f \circ g)'(x) = f'(g(x))g'(x)$

That is, $D_x f(g(x)) = f'(g(x)) \cdot g'(x)$

or

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Chain Rule

Let

$$y = f(u),$$

$$u = \phi(v),$$


$$v = \psi(x),$$

thus the derivative of $y = f\{\phi[\psi(x)]\}$ is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Example 1

? $f(x) = \ln \sin x$, find $f'(x)$

 $f(x) = y = \ln \sin x$


Let $u = \sin x \Rightarrow y = \ln u$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\therefore f'(x) = \frac{1}{u} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x.$$

Example 2

? $f(x) = e^{\sin \frac{1}{x}}$, find $f'(x)$

 $f(x) = y = e^{\sin \frac{1}{x}}$


Let $u = \sin \frac{1}{x}$, $v = \frac{1}{x} \Rightarrow y = e^{u(v)}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$\therefore f'(x) = e^{u(v)} \cdot \cos v \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2} e^{\sin \frac{1}{x}} \cos \frac{1}{x}.$$

Example 3

? $f(x) = (x^2 + 1)^{10}$, find $f'(x)$.

 $f(x) = y = (x^2 + 1)^{10}$

Let $u = x^2 + 1 \Rightarrow y = u^{10}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\therefore f'(x) = 10u^9 \cdot (2x) = 20x(x^2 + 1)^9.$$

Example 4

$$? \quad f(x) = \begin{cases} \frac{\sin^2 x}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad \text{find } f'(x).$$

$$\text{✎ } x \neq 0, \quad f'(x) = \left(\frac{\sin^2 x}{x} \right)' = \frac{\sin 2x}{x} - \frac{\sin^2 x}{x^2},$$

$$x = 0, \quad f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\frac{\sin^2 x}{x} - 0}{x} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} = 1,$$

$$\text{So, } f'(x) = \begin{cases} \frac{\sin 2x}{x} - \frac{\sin^2 x}{x^2}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Example 5



Find $\frac{d}{dx} \frac{1}{(2x-1)^3}$.



$$\begin{aligned} \frac{d}{dx} \frac{1}{(2x-1)^3} &= \frac{d}{dx} (2x-1)^{-3} \\ &= -3(2x-1)^{-3-1} \frac{d}{dx} (2x-1) \\ &= -\frac{6}{(2x-1)^4} \end{aligned}$$

Example 6



Find $D_x \sin[\cos(x^2)]$.



$$D_x \sin[\cos(x^2)]$$

$$= \cos[\cos(x^2)] \cdot [-\sin(x^2)] \cdot 2x$$

$$= -2x \sin(x^2) \cos[\cos(x^2)]$$


Summary of the Chain Rule


The Chain Rules

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Questions and Answers

 $y = \arctan\left(\frac{\sin x}{e^x - 1}\right)$, find y' .


$$y' = \frac{1}{1 + \left(\frac{\sin x}{e^x - 1}\right)^2} \cdot \left(\frac{\sin x}{e^x - 1}\right)'$$

$$= \frac{\cos x(e^x - 1) - e^x \sin x}{(e^x - 1)^2 + \sin^2 x}.$$

Questions and Answers



$$y = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad \text{find } y'$$





$$y' = \left(\frac{x}{2} \sqrt{a^2 - x^2} \right)' + \left(\frac{a^2}{2} \arcsin \frac{x}{a} \right)' \quad (a > 0)$$

$$= \frac{1}{2} \sqrt{a^2 - x^2} + \frac{x}{2} \cdot \frac{1}{2\sqrt{a^2 - x^2}} \cdot (a^2 - x^2)' + \frac{a^2}{2} \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \cdot \left(\frac{x}{a}\right)'$$

$$= \frac{1}{2} \sqrt{a^2 - x^2} - \frac{x^2}{2\sqrt{a^2 - x^2}} + \frac{a^2}{2\sqrt{a^2 - x^2}}.$$

Questions and Answers

 $y = \ln \frac{\sqrt{x^2 + 1}}{\sqrt[3]{x - 2}} \quad (x > 2), \text{ find } y'$

 $\therefore y = \frac{1}{2} \ln(x^2 + 1) - \frac{1}{3} \ln(x - 2)$

$$\therefore y' = \frac{1}{2} \cdot \frac{1}{x^2 + 1} \cdot 2x - \frac{1}{3(x - 2)}$$

$$= \frac{x}{x^2 + 1} - \frac{1}{3(x - 2)}$$

Questions and Answers



If $\phi(x)$ is continuous at $x = a$, $f(x) = (x - a)\phi(x)$, find $f'(a)$.



$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x - a)\phi(x) - 0}{x - a} \\ &= \lim_{x \rightarrow a} \phi(x) \\ &= \phi(a) \end{aligned}$$

Questions and Answers



Find the derivative of $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$



$$\begin{aligned}y' &= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \cdot (x + \sqrt{x + \sqrt{x}})' \\&= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \cdot \left(1 + \frac{1}{2\sqrt{x + \sqrt{x}}} (x + \sqrt{x})'\right) \\&= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \left(1 + \frac{1}{2\sqrt{x + \sqrt{x}}} \left(1 + \frac{1}{2\sqrt{x}}\right)\right) \\&= \frac{4\sqrt{x^2 + x\sqrt{x}} + 2\sqrt{x} + 1}{8\sqrt{x + \sqrt{x + \sqrt{x}}} \cdot \sqrt{x^2 + x\sqrt{x}}}.\end{aligned}$$

The Chain Rule

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